Parametric Performance Analysis of Tracking System using Posterior Cramer Rao Lower Bounds

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This paper analyses the parametric performance of a tracking system using Posterior Cramer Rao Lower Bound (PCRLB). It demonstrates how the influence of model parameters and instrumental signal processing capabilities of tracking filters can be studied using PCRLB without going into specific elaborate design details and Monte Carlo analysis. After briefly reviewing the concept and algorithm of the PCRLB, the lower bound of root mean square error for a bearing only tracking (BOT) has been computed. As in many other tracking problems the process noise covariance matrix ($Q$) is singular in nature in this specific example. So a reformulation of the PCRLB recursion equation is recommended here. The PCRLB performance for position and velocity error has been studied with respect to the variation of (i) the variance of the measurement noise, (ii) the variance of the process noise, and (iii) the sampling time. With this example, it is shown that the PCRLB analysis provides a good handle for tracking system design trade off.

Keywords: Posterior Cramer Rao Lower Bound (PCRLB), Bearing only target tracking

INTRODUCTION

Posterior Cramer Rao Lower Bound (PCRLB) for parameter estimation provides the minimum variance of estimation error that is achievable by any unbiased optimal filter. Although the theory for the same has been around more than half a century but use of it in the field of state estimation of non-linear multi-variable dynamic system has a fairly recent origin mainly due to the work of Taylor and Tichavsky, et al. Such bounds do not have mathematical rigor but has started wide patronage among the target tracking practitioners. Though some have reservation the techniques for providing lower bounds for state estimation of multi-variable dynamic system will be called by the name PCRLB according to Tichavsky, et al.

In linear filtering problem, PCRLB coincides with the error covariance calculated from the Kalman filter algorithm and it has no importance as optimal algorithm exists. For non-linear problem, as no optimal algorithm is available, PCRLB gives an indication of the achievable lower bound of the error covariance and acts as a benchmark to obtain the efficiency of a particular filter. For additive noise models simple algorithms have been published for PCRLB computation. It is noted that, such popular algorithm required either a non-singular process noise covariance matrix ($Q$) or assume it to be a null matrix. It is proposed to use a simpler formulation of the PCRLB recursive equation for a specific restrictive case, which permits singular $Q$ matrix. The reformulation also provides a deeper insight into the variation of error with respect to several parameters. The power of PCRLB analysis is demonstrated with the help of the bearing only target tracking (BOT) problem.

The BOT is used in practical military and civil application including underwater weapon systems, infrared seeker based tracking, sonar based robotic navigation and TV camera or stereo microphone based people tracking. For weapon system guidance, BOT allows use of passive trackers with the consequent tactical advantage. The most important fact is that BOT problems have non-linearity and poor observability problems, which generally stresses the tracking filters. So bearing only tracking (BOT) has been an important field of study and has attracted the attention of many workers. This type of tracking problem is therefore often used for benchmarking tracking and filtering techniques. A typical BOT problem has been described next.
The model parameters of interest are, in order of importance, sampling time, measurement noise covariance, process noise covariance and initial error covariance. It is interesting to know the performance achievable by more powerful signal processing equipment, which permits faster sampling. Similarly, the costly sensors with lower measurement noise covariance may only be justified if it promises substantial performance improvement.

Performances of different types of filters including Extended Kalman Filter (EKF), Pseudo-measurement, uncented KF, and Particle Filter for BOT have been reported in the literature. Some of those studies also explore the effects of varying filter parameters and target tracker parameters. Results of earlier parametric studies sometimes indicated conflicting trends across different families of filters. In the present work, instead of using individual filtering techniques for studying the effects of parameter variation, generic CRPRLB is used.

2D BEARING ONLY TRACKING EXAMPLE

A 2D bearing only tracking example is being presented here to act as a benchmark for CRPRLB studies.

The earliest version of this particular BOT problem occurs in Bar-Shalom and is later re-formulated by Lin et al.

Bearing only tracking requires either (i) multiple tracking stations with known coordinates (where it is a generalised triangulation problem) or (ii) a moving platform with known velocities, on which the tracker is mounted. The non-linear bearing only target-tracking problem used in Lin et al is of the second type. The problem can be visualised by a situation, where a low flying aircraft is tracking an aeroplane or a target aircraft. Though, the numerical data provided in Lin et al is possibly more suitable for an underwater scenario, the low flying aircraft vocabulary is used for appreciation of the problem.

The tracking is done using angle of depression measurement from the airborne platform moving at a nearly fixed altitude. The target is assumed to move in a straight line (assumed to be the X-axis) in the horizontal plane (Figure 1), with a near constant velocity (perturbed by the process noises in position and velocity). The platform, in order to track the target, attempts to move in a course parallel to that of the target with a near constant velocity in the same vertical plane. Imperfections of platform motion are modelled by noise in X and Y directions. The average (neglecting the noisy imperfections) motions of the platform are known with negligible errors. The bearing measurement for the target, however, is noisy. The measurement is processed to estimate the velocity and position of the target in an earth fixed coordinate system. The observer model is non-linear and both process and measurement noises are Gaussian. Note that the platform motion noise would appear as a measurement noise when in an earth fixed frame.

\[ x_{k+1} = F_k x_k + G_k w_k \]  \hspace{1cm} \text{(1)}

with

\[ x_k = \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} \]

\[ F_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \]

and

\[ G_k = \begin{bmatrix} T^2 \\ 2T \end{bmatrix} \]

where \( x_{1,k} \) is the position along X-axis, m; \( x_{2,k} \) the velocity, m/s; \( w_k \) independent zero mean Gaussian white acceleration noise sequence with variance \( q \). The sampling time is denoted as \( T \), which has a nominal value of 1 s. The (unknown) true initial condition and the known noise variance are

\[ x_0 = \begin{bmatrix} 80 \\ 1 \end{bmatrix} \]

\[ q = 0.01 \text{ m}^2/\text{s}^4 \]

The target motion noise covariance matrix may be
computed from above as
\[
Q_k = G_k G_k^T q
\]
\[
= \begin{bmatrix}
\frac{T^2}{2} & \frac{T^2}{2} \\
\frac{T^4}{4} & \frac{T^2}{2}
\end{bmatrix} q
\]
(2)

The tracking platform motion may be described by the following discrete time equations
\[
x_{p,k} = \bar{x}_{p,k} + \Delta x_{p,k} \quad k = 0,1,\ldots,n_{\text{step}}
\]
(3)
\[
y_{p,k} = \bar{y}_{p,k} + \Delta y_{p,k} \quad k = 0,1,\ldots,n_{\text{step}}
\]
(4)
where \(\bar{x}_{p,k}\) and \(\bar{y}_{p,k}\) are the (known) average platform position co-ordinates and \(\Delta x_{p,k}\) and \(\Delta y_{p,k}\) are the mutually independent zero mean Gaussian white noise sequences with variances \(r_x = 1 \times T^2 m^2\) and \(r_y = 1 \times T^2 m^2\), respectively. The mean positions of the platform are
\[\bar{x}_{p,k} = 4kT\]
and
\[\bar{y}_{p,k} = 20\]
The measurement equation (in bearing coordinate) is given as
\[z_{m,k} = z_k + v_{s,k}\]
(5)
where
\[z_k = h\left[\bar{x}_{p,k}, y_{p,k}, x_{1,k}\right]\]
\[= \tan^{-1}\frac{y_{p,k}}{x_{1,k} - x_{p,k}}\]
(6)
is the bearing between the \(X^*\)-axis and the line of sight (LOS) from the sensor to the target and \(v_{s,k}\) is the zero mean Gaussian white measurement noise sequence with variance \(\sigma^2_v = (3\sigma^2)\), assumed to be independent of the sensor platform perturbations and sampling interval.

The random component of platform motion is thus seen to induce additional measurement error, which is non-additive and already embedded in equation (6). The earlier effect can be approximated as additive noise by expanding the nonlinear measurement equations as
\[
z_{m,k} = h\left[\bar{x}_{p,k}, y_{p,k}, x_{1,k}\right] + v_{s,k}
\]
\[= h\left[\bar{x}_{p,k}, \bar{y}_{p,k}, x_{1,k}\right] + v_k\]
(7)
where \(v_k\) is the equivalent additive measurement noise (with variance \(R_k\)) given approximately by small perturbation theory as
\[
v_k = \frac{\bar{y}_{p,k} \Delta x_{p,k} + (x_{1,k} - \bar{x}_{p,k}) \Delta y_{p,k}}{\left[ x_{1,k} - \bar{x}_{p,k} \right]^2 + \bar{y}_{p,k}^2} + v_{s,k}\]
(8)
\(R_k\) is calculated considering \(\Delta x_{p,k}, \Delta y_{p,k}\) and \(v_k\) are mutually independent as
\[R_k = E\left[v_k^2\right] = \frac{\bar{y}_{p,k}^2}{\left[ x_{1,k} - \bar{x}_{p,k} \right]^2 + \bar{y}_{p,k}^2} r_y + r_s\]
(9)
While computation of PCRLB does not require additive noise model, this approximation permits simpler computation of the PCRLB. This also permits comparison of the reported results of several filters using the approximate model.

Filter Initialisation
Traditionally, tracking filters are initialised from first two measurements. The latest bearing measurement defines the initial position estimate and the difference of two bearing measurements provides the estimation of initial velocity.

The initial position estimate thus obtained may be shown to have a covariance of
\[R_{1,0} = \frac{r_x + r_y}{\tan^2 z + \sin^2 z} r_s\]
(10)
The traditional way of initialising velocity estimation would create unduly large variance due to the large measurement error covariance in the present problem. Lin, et al.\(^9\) attempted to reduce such large errors by using prior knowledge about the target motion. As per Lin, et al.\(^9\) the initial velocity estimation is selected as \(\tilde{x}_{2,0} = 0\) and associated variance as \(P_{22,0} = 1\). The off diagonal terms \(P_{12,0}\) and \(P_{21,0}\) are taken as zero. This value of \(P_0\) is referred hereafter as \(P_{\text{nom}}\).
A restriction of the earlier algorithm is that the process excitation covariance matrix \( Q \) must be invertible. This restriction is non-trivial as, tracking problems of the generic type described by equation (1) with scalar noise excitation must necessarily have singular \( Q \) matrix. This however, does not imply that PCRLB is undefined for singular \( Q \) matrix. In fact, PCRLB with a null \( Q \) matrix may be computed \(^1\) from the relations

\[
J_{k+1} = (F_{k}^{-1})^T J_k F_{k}^{-1} + (H_{k+1}^T R_{k+1}^{-1} H_{k+1})
\]

Even recent publications \(^2\) use the two different formulæ for the two situations. A fairly comprehensive review of CRB also does not provide a satisfactory appreciation of the singular \( Q \) matrix problem. The formulation by Tichavsky \(^2\), does mention this problem but provides a very round about way of tackling it.

The difficulties of singular \( Q \) matrix may be avoided by reformulating equation (15). Matrix inversion lemma \(^1\) and the fact that the state matrix is linear. It may be noted that Ristic, et al \(^2\) mentions a similar formula but there are obvious errors in their version. The recursion formula for the information matrix may be reformulated as

\[
J_{k+1} = (Q_k + F_k J_k F_k^T)^{-1} + (H_k^T R_k^{-1} H_k)
\]

Note that, equation (16) has a much simpler form compared to equation (13) and permits singular \( Q \) matrix without any approximation and further, by setting \( Q = 0 \) in equation (16), gives equation (15).

The applicability of this formula, equation (16) has been verified by comparing results with equation (13) for

- Problems with non-singular \( Q \)
- Problems where singular \( Q \) may be perturbed by infinitesimal value to singularity as in the BOT case (typical value 0.001 in diagonal terms).

One would not expect that the RMS error obtained from any other filter to be less than the LBRE obtained from PCRLB, under identical conditions. But this may not be the case always. The cleverer choice of the initial values of the state variable may decrease the RMS error computed from the filters even lower than LBRE for first some steps.

**RESULTS**

**Influence of Measurement Error Covariance**

The effect of substantial changes in measurement error covariance is evident from equation (16). With increase in \( R_k \), \( J_{k+1} \) decreases, thus increasing the LBRE of position and velocity. The reverse is true with decrease in \( R_k \).

Figures 2 and 3 show that increase in measurement error covariance cause poorer settling of position error. When
the measurement error covariance is one-fifth of the nominal value, the terminal position error reduces to 0.48 m compared to 1.01 m in the nominal case. Terminal velocity error reduces to 0.2 m/s compared to 0.29 m/s in the nominal case.

In the sampling time studies in earlier section, the measurement error covariance has been kept constant. However, the measurement error covariance may be coupled with sampling time for some types of passive tracking instruments. A longer sampling interval sometimes provide lower measurement error and this should be kept in mind, while interpreting the result.

**Influence of Process Excitation Covariance**

The effect of changes in process excitation covariance is that with increase in $Q_k$, $J_{k+1}$ decreases as is evident from equation (16). This has the effect of increasing the LBRE of position and velocity.

![Figure 2 Influence of measurement error covariance on LBRE of position](image1)

![Figure 3 Influence of measurement error covariance on LBRE of velocity](image2)

However, it is observed that large variation of $Q$ produces little change in position LBRE.

For velocity LBRE, however, error settling becomes poorer with large error overshoots in velocity as the norm of $Q$ is increased. This implies that there is an overall increase in LBRE with the increase in $Q$.

A larger value of $Q$ indicates larger manoeuvres by the target or greater modelling inaccuracies. The result shows that these may adversely affect the velocity estimation but not so much the position estimation.

**Influence of the Sampling Time on PCRLB**

Equation (16) offers an insight into the effect of sampling time on the PCRLB and thus the LBRE. Substantial decrease in sampling time is expected to cause $J_{k+1}$ to increase, thus decreasing the PCRLB and the LBRE. It is, however, to be noted that the effect of minor changes in sampling time would have to be computed specifically as the change in sampling time would also influence $J_{k+1}$ of the next time step.

The general trend is confirmed in Figures 4 and 5, where the LBRE is calculated and plotted for both position and velocity for different sampling times (0.2 s, 0.5 s, 0.8 s and 1 s). It is seen from the figures that a lower sampling time decreases the PCRLB for both position and velocity. More specifically, while for the nominal sampling time ($T = 1$ s), the time required for the position LBRE to settle to 5 m (about one-fifth of the initial) is about 9 s, the corresponding time for $T = 0.5$ s and $T = 0.2$ s are, respectively 5.5 s and 2.8 s.

Also the effect is more pronounced in initial samples and the differences taper out as the filter converges.

For the velocity LBRE, the time required to settle to 0.6 m/s are, respectively 14 s (for $T = 1$ s), 11.5 s (for $T =$
Figure 5 Influence of the sampling time on LBRE of velocity

0.5 s, 0.2 s (for T = 0.2 s). It may be seen that velocity error does not decay as fast as the position error.

CONCLUSION

Simplified formulation of recursive PCRLB for BOT has been presented in this paper. It provides insight into the influence of parameters. However, for a quantitative understanding of the influence of model parameters and instrumentation/signal processing capabilities on the tracking filter performance PCRLB recursion relations need to be evaluated as shown here.

The PCRLB analysis brings out the fact that large estimation errors may occur from signal processing limitation and tracking instruments and observer kinematics even if the best filter is available.

Specifically, the effects of sampling time and measurement error covariance have been clearly brought out.

So before trying to put efforts in filter designing PCRLB analysis would provide valuable insight.

It should be noted that PCRLB uses identical models for truth and minimum variance estimator. Whereas, filter parameters, such as, initial error covariance matrix filter initialization scheme, choice of process noise covariance are often heuristically changed (keeping the truth model as it is) to obtain performance improvement of tracking filters. So the PCRLB results should not be used as a guide blindly to obtain a cue whether further improvement in the filtering performance is possible or not.

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REFERENCE


